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Group manifold reduction of dual $N = 1$ $d = 10$ supergravity

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Abstract: We perform a group manifold reduction of the dual version of $N = 1$ $d = 10$ supergravity to four dimensions. The effects of the 3- and 4-form gauge fields in the resulting gauged $N = 4$ $d = 4$ supergravity are studied in particular. The example of the group manifold $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is worked out in detail, and we compare for this case the four-dimensional scalar potential with gauged $N = 4$ supergravity.

Keywords: Field Theories in Higher Dimensions, Superstrings and Heterotic Strings, Supergravity Models.

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Contents

1. Introduction

Four-dimensional gauged supergravity theories are candidates for a fundamental description of cosmological inflation. The gauging generates a potential for the scalar fields, which can be compared to the requirements for inflation or for accelerated expansion in general. In this context we have been interested in particular in $N = 4$ supergravity [1, 2]. One reason is that gauged $N = 4$ supergravity can indeed generate a potential with an extremum that corresponds to a positive cosmological constant, as was first shown in [3]. However, a recent analysis showed that the potential of [3] does not satisfy slow-roll conditions [4]. This example is only one of a large class of $N = 4$ models which can be obtained in $N = 4$ supergravity by introducing suitable $\mathrm{SU}(1, 1)$ angles [5]. In [2] all situations where the gauged supergravity is coupled to six additional Yang-Mills multiplets, with all possible (compact and noncompact) semi-simple gauge groups, were analyzed. Although we found examples with a positive extremum of the scalar potential, these cases were not stable against fluctuations of all scalar fields.

The field content studied in [2] can be obtained by dimensional reduction of $N = 1$ $d = 10$ supergravity, which is a second reason for our interest in $N = 4$ supergravity. Reduction over a torus corresponds to the ungauged supergravity theory. Scherk-Schwarz reductions [6] or flux reductions [7] will produce a supergravity in four dimensions in which a non-semisimple group is gauged. However, not all four-dimensional cases can be obtained in this way, since it is not known how to embed the dependence on the SU(1, 1) angles in the reduction procedure. Further versions of $N = 4$ supergravity were obtained in [8] by gauging directly in four dimensions, and in [9, 10] by reduction of IIB supergravity over an orbifold. The relation between these different versions requires further elucidation.

As a first step towards a more complete understanding of gauged $N = 4$ supergravity we consider in this paper the reduction of the dual form [11] of $N = 1$ $d = 10$ supergravity over a compact group manifold. The consistency of this procedure is guaranteed, see [12, 13] for an extensive discussion and references on this topic. We will limit ourselves to the bosonic sector of the theory. The dual version contains a 6-form gauge field as the dual of the 2-form of the standard $N = 1$ theory. The 6-form gauge field reduces to 0-,1-,2-,3- and 4-form gauge fields in four dimensions. In this paper we are particularly interested in the effect of the 3- and 4-forms. In previous reductions of this dual version [11, 14, 15], see also [16] for a review, the role of these forms was not considered in all generality. Our reduction will follow the lines of [17], where a similar but more elaborate analysis of the 2-form reduction was done.

This paper is organized as follows. In section 2 we perform the reduction of the dual $N = 1$ supergravity without additional vector multiplets. The result of this reduction is sufficient for an analysis of the degrees of freedom which arise from the 6-form gauge field. In section 3 we will present a cohomology technique to obtain the massless and massive degrees of freedom for a compact group manifold. In section 4 we extend the results of section 2 by adding additional Yang-Mills fields in ten dimensions. We then analyze the scalar potential in four dimensions of the complete matter coupled system in section 5. In section 6 we will compare with gauged $N = 4$ supergravity in four dimensions.

2. Dual $N = 1$ $D = 10$ supergravity and Scherk-Schwarz reduction

The bosonic fields of $N = 1$ supergravity in ten dimensions are the metric G, the dilaton Φ and the 2-form potential $B^{(2)}$. These can be coupled to Yang-Mills fields A^I, with gauge group $\mathcal{G}_{\rm YM}$. The action in the string frame takes on the form:

$$S_{\text{2-form}} = \int \mathrm{e}^{-\Phi}\left(R\star\mathbb{1} + \star\mathrm{d}\Phi\wedge\mathrm{d}\Phi - \frac{1}{2}\star H^{(3)}\wedge H^{(3)} - \mathrm{Tr}\star F(A)\wedge F(A)\right). \tag{2.1}$$

The gauge invariant three-form field strength reads

$$H^{(3)} = \mathrm{d}B^{(2)} - \mathrm{Tr}\left(A\wedge\mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right). \tag{2.2}$$

The generators of $\mathcal{G}_{\rm YM}$ satisfy $\mathrm{Tr}\, T_I T_J = \frac{1}{2}\delta_{IJ}$.

The two-form potential $B^{(2)}$ can be dualized to a six-form $B^{(6)}$ [11]. The action (2.1) then becomes

$$
\begin{aligned}
S_{\text{6-form}} &= S_{\text{SG}} + S_{\text{YM}}\,, &\text{(2.3a)}\\
S_{\text{SG}} &= \int \Big(\mathrm{e}^{-\Phi}(R\star\mathbb{1} + \star\mathrm{d}\Phi\wedge\mathrm{d}\Phi) - \frac{1}{2}\mathrm{e}^{\Phi}\star H^{(7)}\wedge H^{(7)}\Big)\,, &\text{(2.3b)}\\
S_{\text{YM}} &= -\int \Big(\mathrm{e}^{-\Phi}\,\mathrm{Tr}\star F(A)\wedge F(A) - B^{(6)}\wedge\mathrm{Tr}\,F(A)\wedge F(A)\Big)\,, &\text{(2.3c)}
\end{aligned}
$$

where $H^{(7)} = \mathrm{d}B^{(6)}$. The gauge transformations of $B^{(6)}$ are $\delta B^{(6)} = \mathrm{d}\Lambda^{(5)}$.

We reduce the action (2.3a) over a compact group $\mathcal{G}$ [6]. In this section we will limit ourselves to the supergravity part (2.3b). We will analyze the result of this reduction in section 3. In section 4 we will reduce the contribution of the Yang-Mills fields (2.3c).

The Ansatz for the vielbein is.[1]

$$
\hat{e}^a = e^a\,, \qquad \hat{e}^m = E^m_\alpha(\sigma^\alpha - V^\alpha) \equiv E^m_\alpha\hat{e}^\alpha\,. \tag{2.4}
$$

The $V^\alpha = V^\alpha_\mu\mathrm{d}x^\mu$ are the Kaluza-Klein gauge fields, and E^m_α are scalars which make up the internal metric $G_{\alpha\beta} = \delta_{mn}E^m_\alpha E^n_\beta$. The σ^α are the left-invariant Maurer-Cartan 1-forms of the internal group.[2] They satisfy the Maurer-Cartan relation

$$
\mathrm{d}\sigma^\alpha = -\frac{1}{2}f^\alpha{}_{\beta\gamma}\sigma^\beta\wedge\sigma^\gamma\,. \tag{2.5}
$$

Our parametrization of $\hat{e}^\alpha$ in (2.4) leads to

$$
\mathrm{d}\hat{e}^\alpha = -F(V)^\alpha - f^\alpha{}_{\beta\gamma}\hat{e}^\beta\wedge V^\gamma - \frac{1}{2}f^\alpha{}_{\beta\gamma}\hat{e}^\beta\wedge\hat{e}^\gamma\,. \tag{2.6}
$$

$F(V)$, the field strength of the Kaluza-Klein vectors, is given explicitly by

$$
F(V)^\alpha = \mathrm{d}V^\alpha + [V,V]^\alpha = \mathrm{d}V^\alpha + \frac{1}{2}f^\alpha{}_{\beta\gamma}V^\beta\wedge V^\gamma\,. \tag{2.7}
$$

The reduction Ansätze for the other fields are given by:[3]

$$
\begin{aligned}
\hat{\Phi} &= \phi + \frac{1}{2}\ln|\det G_{\alpha\beta}|\,, &\text{(2.8)}\\
\hat{B}^{(6)} &= \frac{1}{2!}B^{(4)}{}_{\alpha_1\alpha_2}\,\hat{e}^{\alpha_1}\hat{e}^{\alpha_2} + \frac{1}{3!}B^{(3)}{}_{\alpha_1\ldots\alpha_3}\hat{e}^{\alpha_1}\cdots\hat{e}^{\alpha_3} + \frac{1}{4!}B^{(2)}{}_{\alpha_1\ldots\alpha_4}\hat{e}^{\alpha_1}\cdots\hat{e}^{\alpha_4} + \\
&\quad + \frac{1}{5!}B^{(1)}{}_{\alpha_1\ldots\alpha_5}\hat{e}^{\alpha_1}\cdots\hat{e}^{\alpha_5} + \frac{1}{6!}B^{(0)}{}_{\alpha_1\ldots\alpha_6}\hat{e}^{\alpha_1}\cdots\hat{e}^{\alpha_6}\,. &\text{(2.9)}
\end{aligned}
$$

For the m-forms $B^{(m)}$ we used the shorthand notation

$$
B^{(m)}{}_{\alpha_1\ldots\alpha_{6-m}} = \frac{1}{m!}B^{(m)}{}_{a_1\ldots a_m\alpha_1\ldots\alpha_{6-m}}e^{a_1}\cdots e^{a_m}\,, \qquad m = 0,1,\ldots,6\,. \tag{2.10}
$$

[1] From now on ten-dimensional fields and indices will be indicated by a hat. The tangent space indices are split up as $\hat{a} = (a,m)$ where a runs over $1,\ldots,4$ and $m = 1,\ldots,6$, and the curved indices as $\hat{\mu} = (\mu,\alpha)$ where $\mu = 1,\ldots,4$ and $\alpha = 1,\ldots,6$. Further notational matters are discussed in appendix A.

[2] The volume of $\mathcal{G}$ is normalized as $\int_{\mathcal{G}}\sigma^1\wedge\ldots\wedge\sigma^6 = 1$. The compactness of $\mathcal{G}$ implies that the structure constants are traceless, $f^\alpha{}_{\alpha\beta} = 0$.

[3] We will omit the $\wedge$-symbols from now on to simplify the formulae.

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From the Ansatz (2.9) one obtains the field strength via $\hat{H}^{(7)} = \mathrm{d}\hat{B}^{(6)}$. By substituting the expression for $\hat{B}$ this expands to:

$$\hat{H}^{(7)} = \sum_{m=1}^{4} \frac{1}{m!(7-m)!} H^{(m)}{}_{a_1 \dots a_m \alpha_1 \dots \alpha_{7-m}} e^{a_1} \cdots e^{a_m} \hat{e}^{\alpha_1} \cdots \hat{e}^{\alpha_{7-m}} \,. \tag{2.11}$$

The reduction of the action (2.3b) gives the following four-dimensional result:

$$S = S_1 + S_2 \,, \tag{2.12a}$$

$$S_1 = \int \mathrm{e}^{-\phi} \Big(R \star\mathbb{1} + \frac{1}{2} \star\mathrm{d}\phi\, \mathrm{d}\phi - \frac{1}{2} G_{\alpha\beta} \star F(V)^\alpha F(V)^\beta + \\ + \frac{1}{4} \star D G_{\alpha\beta} D G^{\alpha\beta} - V_1(G) \star\mathbb{1} \Big) \,, \tag{2.12b}$$

$$S_2 = -\frac{1}{2} \int \mathrm{e}^{\phi} \, \det G_{\alpha\beta} \Big(\frac{1}{3!} \star H^{(4)\alpha_1 \dots \alpha_3} H^{(4)}{}_{\alpha_1 \dots \alpha_3} + \frac{1}{4!} \star H^{(3)\alpha_1 \dots \alpha_4} H^{(3)}{}_{\alpha_1 \dots \alpha_4} + \\ + \frac{1}{5!} \star H^{(2)\alpha_1 \dots \alpha_5} H^{(2)}{}_{\alpha_1 \dots \alpha_5} + \frac{1}{6!} \star H^{(1)\alpha_1 \dots \alpha_6} H^{(1)}{}_{\alpha_1 \dots \alpha_6} \Big) \,. \tag{2.12c}$$

D is covariant with respect to Scherk-Schwarz gauge transformations. The contraction in the H^2-terms is with $G^{\alpha\beta}$. The scalar potential $V_1(G)$ due to reduction of the gravitational sector is given by [6]

$$V_1(G) = \frac{1}{4} G_{\alpha_1\alpha_2} G^{\beta_1\beta_2} G^{\gamma_1\gamma_2} f^{\alpha_1}{}_{\beta_1\gamma_1} f^{\alpha_2}{}_{\beta_2\gamma_2} + \frac{1}{2} G^{\alpha_1\alpha_2} f^{\beta}{}_{\alpha_1\gamma} f^{\gamma}{}_{\alpha_2\beta} \,. \tag{2.13}$$

The explicit form of the curvatures $H^{(p)}$ is

$$H^{(1)}{}_{\alpha_1 \dots \alpha_6} = D B^{(0)}{}_{\alpha_1 \dots \alpha_6} + 15 f^{\beta}{}_{[\alpha_1\alpha_2} B^{(1)}{}_{\alpha_3 \dots \alpha_6]\beta} \,, \tag{2.14a}$$

$$H^{(2)}{}_{\alpha_1 \dots \alpha_5} = D B^{(1)}{}_{\alpha_1 \dots \alpha_5} + B^{(0)}_{\alpha_1 \dots \alpha_5 \beta} F(V)^\beta + 10 f^{\beta}{}_{[\alpha_1\alpha_2} B^{(2)}{}_{\alpha_3 \dots \alpha_5]\beta} \,, \tag{2.14b}$$

$$H^{(3)}{}_{\alpha_1 \dots \alpha_4} = D B^{(2)}{}_{\alpha_1 \dots \alpha_4} + B^{(1)}{}_{\alpha_1 \dots \alpha_4 \beta} F(V)^\beta + 6 f^{\beta}{}_{[\alpha_1\alpha_2} B^{(3)}{}_{\alpha_3\alpha_4]\beta} \,, \tag{2.14c}$$

$$H^{(4)}{}_{\alpha_1 \dots \alpha_3} = D B^{(3)}{}_{\alpha_1 \dots \alpha_3} + B^{(2)}{}_{\alpha_1 \dots \alpha_3 \beta} F(V)^\beta + 3 f^{\beta}{}_{[\alpha_1\alpha_2} B^{(4)}{}_{\alpha_3]\beta} \,. \tag{2.14d}$$

The gauge transformations of $B^{(n)}$ are

$$\delta B^{(n)}{}_{\alpha_1 \dots \alpha_{6-n}} = D \Lambda^{(n-1)}{}_{\alpha_1 \dots \alpha_{6-n}} - \Lambda^{(n-2)}{}_{\alpha_1 \dots \alpha_{6-n}\beta} F(V)^\beta - \\ - \frac{1}{2} (6-n)(5-n) f^{\beta}{}_{[\alpha_1\alpha_2} \Lambda^{(n)}{}_{\alpha_3 \dots \alpha_{6-n}]\beta} \,. \tag{2.15}$$

The formulation using lower internal indices is particularly useful in our cohomology analysis of the Kalb-Ramond degrees of freedom in sections 3.1 and appendix B. For other aspects of the group manifold reduction it is more convenient to redefine the p-form gauge fields by dualizing to upper internal indices:

$$B^{(n)}{}_{\alpha_1 \dots \alpha_{6-n}} \equiv \frac{1}{n!} \tilde{\varepsilon}_{\alpha_1 \dots \alpha_{6-n}\, \beta_1 \dots \beta_n} \tilde{B}^{(n)\beta_1 \dots \beta_n} \,. \tag{2.16}$$

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The corresponding curvatures are:

$$\begin{aligned}\tilde{H}^{(n)\alpha_1\dots\alpha_{n-1}} &= D\tilde{B}^{(n-1)\alpha_1\dots\alpha_{n-1}} + (n-1)(-1)^n\,\tilde{B}^{(n-2)[\alpha_1\dots\alpha_{n-2}}F(V)^{\alpha_{n-1}]} + \\ &\quad + \frac{1}{2}(n-1)\,f^{[\alpha_1}{}_{\beta_1\beta_2}\tilde{B}^{(n)\alpha_2\dots\alpha_{n-1}]\beta_1\beta_2}\,. \end{aligned} \tag{2.17}$$

The reduced gauge transformations of the $\tilde{B}^{(n)}$ are

$$\delta\tilde{B}^{(4)\alpha_1\dots\alpha_4} = D\tilde{\Lambda}^{(3)\alpha_1\dots\alpha_4} - 4F^{[\alpha_1}(V)\tilde{\Lambda}^{(2)\alpha_2\alpha_3\alpha_4]} - 2f^{[\alpha_1}{}_{\beta_1\beta_2}\tilde{\Lambda}^{(4)\alpha_2\alpha_3\alpha_4]\beta_1\beta_2}\,, \tag{2.18a}$$

$$\delta\tilde{B}^{(3)\alpha_1\alpha_2\alpha_3} = D\tilde{\Lambda}^{(2)\alpha_1\alpha_2\alpha_3} - 3F^{[\alpha_1}(V)\tilde{\Lambda}^{(1)\alpha_2\alpha_3]} - \frac{3}{2}f^{[\alpha_1}{}_{\beta_1\beta_2}\tilde{\Lambda}^{(3)\alpha_2\alpha_3]\beta_1\beta_2}\,, \tag{2.18b}$$

$$\delta\tilde{B}^{(2)\alpha_1\alpha_2} = D\tilde{\Lambda}^{(1)\alpha_1\alpha_2} - 2F^{[\alpha_1}(V)\tilde{\Lambda}^{(0)\alpha_2]} - f^{[\alpha_1}{}_{\beta_1\beta_2}\tilde{\Lambda}^{(2)\alpha_2]\beta_1\beta_2}\,, \tag{2.18c}$$

$$\delta B^{(1)\alpha_1} = D\tilde{\Lambda}^{(0)\alpha_1} - \frac{1}{2}f^{\alpha_1}{}_{\beta_1\beta_2}\tilde{\Lambda}^{(1)\beta_1\beta_2}\,, \tag{2.18d}$$

$$\delta\tilde{B}^{(0)} = 0\,. \tag{2.18e}$$

Finally we transform the action (2.12a) to the Einstein frame, and express the result in terms of the curvatures (2.17). We find

$$S = S_1 + S_2\,, \tag{2.19a}$$

$$\begin{aligned} S_1 &= \int\Bigg(R\star\mathbb{1} - \frac{1}{2}\star\mathrm{d}\phi\,\mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{-\phi}\,G_{\alpha\beta}\star F(V)^\alpha F(V)^\beta + \\ &\quad + \frac{1}{4}\star DG_{\alpha\beta}DG^{\alpha\beta} - \mathrm{e}^{\phi}\,V_1(G)\star\mathbb{1}\Bigg)\,, \end{aligned} \tag{2.19b}$$

$$\begin{aligned} S_2 &= -\frac{1}{2}\int\Bigg(\frac{1}{3!}\mathrm{e}^{-\phi}\star\tilde{H}^{(4)\alpha_1\dots\alpha_3}\tilde{H}^{(4)}{}_{\alpha_1\dots\alpha_3} + \frac{1}{2}\star\tilde{H}^{(3)\alpha_1\alpha_2}\tilde{H}^{(3)}{}_{\alpha_1\alpha_2} + \\ &\quad + \mathrm{e}^{\phi}\star\tilde{H}^{(2)\alpha_1}\tilde{H}^{(2)}{}_{\alpha_1} + \mathrm{e}^{2\phi}\star\tilde{H}^{(1)}\tilde{H}^{(1)}\Bigg)\,, \end{aligned} \tag{2.19c}$$

where the contraction in the $\tilde{H}^2$-terms is with $G_{\alpha\beta}$.

3. Analysis in $D = 4$

The massless 3-form fields and the 4-forms that are obtained from the reduction of $\hat{B}^{(6)}$ do not carry physical degrees of freedom. The 3-forms however do modify the scalar potential through their vacuum expectation values (vevs). The first part of this section is therefore devoted to an analysis of the shift symmetries $\delta B^{(n)} \sim f\Lambda^{(n)}$ in (2.15) in order to determine the (non)physical components of the $B^{(n)}$ and their masses, which arise through similar terms $H^{(n)} \sim fB^{(n)}$ in the curvatures (2.14). Section 3.1 treats the general case, in section 3.2 we discuss the example of $\mathcal{G} = \mathrm{SU}(2)\times\mathrm{SU}(2)$.

3.1 Cohomology of Scherk-Schwarz gauge transformations

For a generic field Φ with p internal indices we have a local shift symmetry

$$\delta\Phi_{\alpha_1\dots\alpha_p} = f^{\beta}{}_{[\alpha_1\alpha_2}\Lambda_{\alpha_3\dots\alpha_p]\beta}\,, \tag{3.1}$$

a mass term for this field is constructed from

$$(m[\Phi])_{\alpha_1\ldots\alpha_{p+1}} = f^{\beta}{}_{[\alpha_1\alpha_2}\Phi_{\alpha_3\ldots\alpha_{p+1}]\beta}\,. \tag{3.2}$$

We associate with each Φ an element of the set $\Omega^{(p)}$ of left-invariant p-forms on $\mathcal{G}$:

$$\Phi_{\alpha_1\ldots\alpha_p} \mapsto \Phi^{(p)} = \frac{1}{p!}\Phi_{\alpha_1\ldots\alpha_p}\sigma^{\alpha_1}\ldots\sigma^{\alpha_p}\,. \tag{3.3}$$

Given the exterior derivative $\mathrm{d}_p : \Omega^{(p)} \to \Omega^{(p+1)}$, we define its image $C^{(p)} \equiv \mathrm{Im}\,(\mathrm{d}_{p-1})$, its kernel $Z^{(p)} = \mathrm{Ker}\,(\mathrm{d}_p)$ and the quotient $H^{(p)} = Z^{(p)}/C^{(p)}$. The equations (3.1), (3.2) are equivalent to a Rham cohomology problem for the left-invariant forms:

$$\delta\Phi^{(p)} = \mathrm{d}\Lambda^{(p-1)}\,, \qquad m[\Phi^{(p)}] = \mathrm{d}\Phi^{(p)}\,. \tag{3.4}$$

A well-known theorem by Chevalley and Eilenberg [18] tells us that this cohomology problem is equivalent to the usual de Rham cohomology. In particular

$$\dim H^{(p)} = b_p\,, \tag{3.5}$$

where b_p is the pth Betti number of $\mathcal{G}$. Since d_p is a homomorphism we have

$$C^{(p)} \cong \Omega^{(p-1)}/Z^{(p-1)}\,, \tag{3.6}$$

giving us the recurrence relation

$$\dim C^{(p+1)} = \dim\Omega^{(p)} - \dim C^{(p)} - b_p = \begin{pmatrix} 6 \\ p \end{pmatrix} - b_p - \dim C^{(p)}\,. \tag{3.7}$$

This recurrence relation can be solved starting from $\dim C^{(0)} = 0$:

$$\begin{aligned}
\dim C^{(0)} &= 0\\
\dim C^{(1)} &= 1 - b_0\\
\dim C^{(2)} &= 5 - b_1 + b_0\\
\dim C^{(3)} &= 10 - b_2 + b_1 - b_0\\
\dim C^{(4)} &= 10 - b_3 + b_2 - b_1 + b_0\\
\dim C^{(5)} &= 5 - b_4 + b_3 - b_2 + b_1 - b_0 = 5 - \chi(\mathcal{G}) + b_6 - b_5\\
\dim C^{(6)} &= 1 - b_5 + b_4 - b_3 + b_2 - b_1 + b_0 = 1 + \chi(\mathcal{G}) - b_6\,,
\end{aligned} \tag{3.8}$$

where $\chi(\mathcal{G})$ is the Euler characteristic

$$\chi(\mathcal{G}) = \sum_{r=0}^{6}(-1)^r b_r\,. \tag{3.9}$$

We draw the following conclusions:

– The mass terms are invariant under the shift symmetries.

field	features	dimension	# DOF
$B^{(4)}$	gauge	$\dim C^{(2)}$	0
	massive	$\dim C^{(3)}$	0
	massless	b_2	0
$B^{(3)}$	gauge	$\dim C^{(3)}$	0
	massive	$\dim C^{(4)}$	1
	massless	b_3	0
$B^{(2)}$	gauge	$\dim C^{(4)}$	0
	massive	$\dim C^{(5)}$	3
	massless	b_4	1
$B^{(1)}$	gauge	$\dim C^{(5)}$	0
	massive	$\dim C^{(6)}$	3
	massless	b_5	2
$B^{(0)}$	gauge	$\dim C^{(6)}$	0
	massive	0	1
	massless	b_6	1

Table 1: Result of the analysis of the Scherk-Schwarz gauge transformations of the Kalb-Ramond fields $B^{(n)}$. Each $B^{(n)}$ splits in three parts: gauge degrees of freedom, massive and massless components. The third column shows how the dimension of the various spaces $C^{(p)}$ and $H^{(p)}$ determines the number of components of $B^{(n)}$. DOF indicates the number of spacetime degrees of freedom for each choice of the $6-n$ internal indices. Hence the total number of degrees of freedom per field is obtained by multiplying horizontally.

– A field can be gauged away if $\Phi^{(p)} \in C^{(p)}$. These fields do not have mass terms.

– If and only if a field $\Phi^{(p)}$ has a mass term, there is a $\Phi^{(p+1)}$ that can be gauged away, i.e. $\Phi^{(p+1)}$ is 'eaten' by $\Phi^{(p)}$.

– Physical massless fields are elements of $H^{(p)}$, the number of these fields is b_p.

– The number of massive fields in $\Omega^{(p)}$ is $\dim C^{(p+1)}$.

In this context the fields $B^{(n)}$ in four dimensions are interpreted as elements of $\Omega^{(6-n)}$, for $n = 0, \ldots, 4$. The implications for these fields are presented in table 1. The total number of physical degrees of freedom in four dimensions is obtained by taking, for each row of table 1, the product of the dimension in the third, and the number of degrees of freedom in the fourth column, and by summing these products. Using (3.8) one finds that this sum is $28 + \chi(\mathcal{G})$. For a compact connected group, $\chi(\mathcal{G}) = 0$ (see e.g. [19]); we thus recover the 28 degrees of freedom of the ten-dimensional $\hat{B}^{(6)}$.

We conclude from the above cohomology analysis that we can give a vev to a $B^{(3)}$ if $b_3 \neq 0$. This is always the case, since for any compact nonabelian group there is a nonzero harmonic 3-form

$$\Omega_3 = \mathrm{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)\,, \qquad g \in \mathcal{G}\,. \tag{3.10}$$

field	features	dimension	# DOF	total DOF
$B^{(4)}$	gauge	6	0	0
	massive	9	0	0
	massless	0	0	0
$B^{(3)}$	gauge	9	0	0
	massive	9	1	9
	massless	2	0	0
$B^{(2)}$	gauge	9	0	0
	massive	6	3	18
	massless	0	1	0
$B^{(1)}$	gauge	6	0	0
	massive	0	3	0
	massless	0	2	0
$B^{(0)}$	gauge	0	0	0
	massive	0	1	0
	massless	1	1	1

Table 2: Result of the analysis of the Scherk-Schwarz gauge transformations of the Kalb-Ramond fields $B^{(n)}$ for the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$, in the reduction of $B^{(6)}$ from 10 to 4 dimensions. See the caption of table 1 for more details. In the last column we indicate the total numbers of degrees of freedom.

In the case of $B^{(4)}$ the equation of motion is algebraic, it sets, in the absence of Yang-Mills fields, certain components of $H^{(4)}$ to zero. The b_2 massless components in table 1 do not appear in the action at all, and never give rise to fluxes.

If $\mathcal{G}$ is semi-simple, $b_1 = b_5 = 0$. Combining this with $b_0 = b_6 = 1$ and $\chi(\mathcal{G}) = 0$, we see that all vectors $B^{(1)}$ can be gauged away.

Explicit examples of the analysis for specific group manifolds are given in the following subsection, and in appendix B.

3.2 Analysis for $\mathrm{SU}(2) \times \mathrm{SU}(2)$

In this subsection we will work out the example of the gauge group $\mathrm{SU}(2) \times \mathrm{SU}(2)$. We first give the results following the methods of section 3.1. Then we rederive the results by analyzing the Scherk-Schwarz gauge transformations explicitly.

For $\mathrm{SU}(2) \times \mathrm{SU}(2)$ the Betti numbers are given by

$$G = \mathrm{SU}(2) \times \mathrm{SU}(2) \Rightarrow (b_0, \ldots, b_6) = (1, 0, 0, 2, 0, 0, 1)\,. \tag{3.11}$$

The resulting degrees of freedom are given in table 2. $B^{(2)}$ and $B^{(3)}$ give only massive degrees of freedom, which are therefore stabilized. The single massless degree of freedom corresponds to the axion.

In our explicit analysis we indicate the six internal directions by indices $a, b, \ldots = 1, 2, 3$ and $\bar{a}, \bar{b}, \ldots = 4, 5, 6$, with one $\mathrm{SU}(2)$ gauging the $a, b, \ldots$, and the other the $\bar{a}, \bar{b}, \ldots$ directions. In this analysis, which uses (2.17), (2.18), we first investigate which field components of $\tilde{B}^{(n)}$ can be gauged away by the shift transformations involving $\tilde{\Lambda}^{(n)}$, and then look at the impact of this process on the equations of motion.

To start, all six vectors $\tilde{B}^{(1)}$ can be gauged away by the transformations

$$\delta\tilde{B}^{(1)a} = -\frac{1}{2} f^{a}{}_{cd}\tilde{\Lambda}^{(1)cd}\,, \qquad \delta\tilde{B}^{(1)\bar{a}} = -\frac{1}{2} f^{\bar{a}}{}_{\bar{c}\bar{d}}\tilde{\Lambda}^{(1)\bar{c}\bar{d}}\,. \tag{3.12}$$

Note that this does not involve the parameters $\Lambda^{a\bar{a}}$, which therefore remain free. The vectors which we gauge away will contribute to masses for $\tilde{B}^{(2)ab}$ and $\tilde{B}^{(2)\bar{a}\bar{b}}$. These mass terms are invariant under the shift transformations of $\tilde{B}^{(2)}$, due to the Jacobi identity for the structure constants. The remaining fields $\tilde{B}^{(2)a\bar{a}}$ transform as

$$\delta\tilde{B}^{(2)a\bar{a}} = -\Big(f^{a}{}_{cd}\tilde{\Lambda}^{(2)\bar{a}cd} - f^{\bar{a}}{}_{\bar{c}\bar{d}}\tilde{\Lambda}^{(2)a\bar{c}\bar{d}}\Big)\,. \tag{3.13}$$

This implies that these components of $B^{(2)}$ can be gauged away. The components

$$\tilde{B}^{(3)a\bar{a}}_{A} = f^{a}{}_{cd}\tilde{B}^{(3)\bar{a}cd} - f^{\bar{a}}{}_{\bar{c}\bar{d}}\tilde{B}^{(3)a\bar{c}\bar{d}} \tag{3.14}$$

then become massive. All this implies that 6 vectors are gauged away, there are 6 massive 2-forms corresponding to 18 degrees of freedom, and 9 massive 3-forms, which have 9 degrees of freedom. Together with the axion $\tilde{B}^{(0)}$ we find the 28 degrees of freedom of $\hat{B}^{(6)}$.

For $\tilde{B}^{(3)}$ the remaining components are $\tilde{B}^{(3)abc}$ and $\tilde{B}^{(3)\bar{a}\bar{b}\bar{c}}$, which are both gauge invariant under SU(2)$\times$SU(2) and cannot be gauged away, and the symmetric combination

$$\tilde{B}^{(3)a\bar{a}}_{S} = f^{a}{}_{cd}\tilde{B}^{(3)\bar{a}cd} + f^{\bar{a}}{}_{\bar{c}\bar{d}}\tilde{B}^{(3)a\bar{c}\bar{d}}\,. \tag{3.15}$$

Under the shifts of $\tilde{B}^{(3)}$ this transforms as

$$\delta\tilde{B}^{(3)a\bar{a}}_{S} = -3 f^{a}{}_{cd} f^{\bar{a}}{}_{\bar{c}\bar{d}}\tilde{\Lambda}^{(3)cd\bar{c}\bar{d}}\,, \tag{3.16}$$

so that these symmetric components can be gauged away. This implies that the terms involving $\tilde{B}^{(4)ab\bar{a}\bar{b}}$ remain in the action. In fact, they are (for the chosen gauge group) the only components of $\tilde{B}^{(4)}$ which appear in the action in the first place.

The other components of $\tilde{B}^{(3)}$ and those of $\tilde{B}^{(4)}$ can be treated as follows. The components $\tilde{B}^{(3)abc}$ and $\tilde{B}^{(3)\bar{a}\bar{b}\bar{c}}$ are gauge invariant, and they do not appear in $\tilde{H}^{(3)}$. Their field equation is therefore simply:

$$D\big(\mathrm{e}^{-\phi}\star\tilde{H}^{(4)}{}_{abc}\big) = 0\,, \qquad D\big(\mathrm{e}^{-\phi}\star\tilde{H}^{(4)}{}_{\bar{a}\bar{b}\bar{c}}\big) = 0\,, \tag{3.17}$$

where internal indices have been lowered with the internal metric. We conclude that these components of $\tilde{H}^{(4)}$ give two independent constants: they are the $\tilde{B}^{(3)}$ vevs discussed in section 3.1. These vevs are precisely those given by (3.10) for the two SU(2)-factors:

$$\tilde{H}^{\text{vev}}_{(4)} \sim m_1\,\varepsilon_{abc}\,\sigma^{a}\sigma^{b}\sigma^{c} + m_2\,\varepsilon_{\bar{a}\bar{b}\bar{c}}\,\sigma^{\bar{a}}\sigma^{\bar{b}}\sigma^{\bar{c}}\,. \tag{3.18}$$

Of course in this analysis we ignore the interaction with other fields, we will come back to the complete equations of motion in section 5.

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The equation of motion of the remaining $\tilde{B}^{(4)}$ components is

$$\tilde{H}^{(4)}{}_{a\bar{c}\bar{d}} f^{a}{}_{cd} + \tilde{H}^{(4)}{}_{\bar{a}cd} f^{\bar{a}}{}_{\bar{c}\bar{d}} = 0\,, \tag{3.19}$$

thus putting the field strengths of the unphysical symmetric combination $\tilde{B}_S^{(3)a\bar{a}}$ to zero.

Therefore we see that the propagating degrees of freedom in this case come from $\tilde{B}^{(2)ab}$ and $\tilde{B}^{(2)\bar{a}\bar{b}}$, and the $\tilde{B}_A^{(3)a\bar{a}}$ (3.14). This of course agrees with the cohomology analysis in section 3.1. The nonpropagating degrees of freedom $\tilde{B}^{(3)abc}$ and $\tilde{B}^{(3)\bar{a}\bar{b}\bar{c}}$ lead to constant components of $\tilde{H}^{(4)}$ which contribute to the scalar potential.

4. Coupling to Yang-Mills multiplets

In this section we reduce the Yang-Mills sector (2.3c). The Ansatz for the ten-dimensional Yang-Mills field is

$$\hat{A}^I = A^I{}_a e^a + A^I{}_\alpha \hat{e}^\alpha\,. \tag{4.1}$$

The field strength can then be obtained from $\hat{F}(\hat{A})^I = d\hat{A}^I + (\hat{A}\wedge\hat{A})^I$ with the result:[4]

$$\hat{F}(\hat{A})^I = G^I + \bar{D}A^I{}_\alpha e^\alpha + \frac{1}{2}\mathcal{F}^I{}_{\alpha_1\alpha_2}\,\hat{e}^{\alpha_1}\hat{e}^{\alpha_2}\,, \tag{4.2}$$

where we use the following definitions:

$$F(A)^I = \mathrm{d}A^I + \frac{1}{2}f^I{}_{JK}A^J A^K\,, \tag{4.3a}$$

$$G^I = F(A)^I - A^I{}_\alpha F^\alpha(V)\,, \tag{4.3b}$$

$$\bar{D}A^I{}_\alpha = \mathrm{d}A^I{}_\alpha + f^\beta{}_{\alpha\gamma}V^\gamma A^I{}_\beta + f^I{}_{JK}A^J A^K{}_\alpha\,, \tag{4.3c}$$

$$\mathcal{F}^I{}_{\alpha_1\alpha_2} = f^I{}_{JK}A^J{}_{\alpha_1}A^K{}_{\alpha_2} - A^I{}_\gamma f^\gamma{}_{\alpha_1\alpha_2}\,. \tag{4.3d}$$

The ten-dimensional Chern-Simons form $\hat{C}$ can be rewritten as

$$\hat{C} = \mathrm{Tr}\left(\hat{A}\,\mathrm{d}\hat{A} + \frac{2}{3}\hat{A}\,\hat{A}\,\hat{A}\right) = \frac{1}{2}\hat{A}^I\,\hat{F}(\hat{A})^I - \frac{1}{12}f^I{}_{JK}\hat{A}^I\,\hat{A}^J\,\hat{A}^K\,. \tag{4.4}$$

Substitution of the Ansatz (4.1) gives

$$\hat{C} = C^{(3)} + C^{(2)}{}_\alpha\hat{e}^\alpha + \frac{1}{2!}C^{(1)}{}_{\alpha_1\alpha_2}\hat{e}^{\alpha_1}\hat{e}^{\alpha_2} + \frac{1}{3!}C^{(0)}{}_{\alpha_1\alpha_2\alpha_3}\hat{e}^{\alpha_1}\hat{e}^{\alpha_2}\hat{e}^{\alpha_3}\,, \tag{4.5}$$

where

$$C^{(3)} = \frac{1}{2}\left(A^I G^I - \frac{1}{6}f^I{}_{JK}A^I A^J A^K\right), \tag{4.6a}$$

$$C^{(2)}{}_\alpha = \frac{1}{2}\left(A^I\bar{D}A^I{}_\alpha + A^I{}_\alpha G^I - \frac{1}{2}f^I{}_{JK}A^I A^J A^K{}_\alpha\right), \tag{4.6b}$$

[4]In the following $\bar{D}$ stands for a covariant derivative which is covariant with respect to both the Scherk-Schwarz and Yang-Mills gauge transformations. Again we omit the $\wedge$-symbols. It should be kept in mind that A^I is the one-form corresponding to the gauge vector, while $A^I{}_\alpha$ are scalar fields. The internal indices $\alpha, \beta, \ldots$ are therefore always written explicitly.

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$$C^{(1)}{}_{\alpha_1\alpha_2} = -\frac{1}{2}A^I A^I{}_\gamma f^\gamma{}_{\alpha_1\alpha_2} - A^I{}_{[\alpha_1}\bar{D}A^I{}_{\alpha_2]}\,, \tag{4.6c}$$

$$C^{(0)}{}_{\alpha_1\alpha_2\alpha_3} = f^I{}_{JK}A^I{}_{\alpha_1}A^J{}_{\alpha_2}A^K{}_{\alpha_3} - \frac{3}{2}A^I{}_{[\alpha_1}f^\delta{}_{\alpha_2\alpha_3]}A^I{}_\delta\,. \tag{4.6d}$$

The reduction of the ten-dimensional action (2.3c) is facilitated by using $\mathrm{d}\hat{C} = \frac{1}{2}\hat{F}^I\hat{F}^I$ to write

$$\mathcal{L}_{\mathrm{CS}} = \hat{B}^{(6)} \wedge \mathrm{Tr}\,(\hat{F}\wedge\hat{F}) = -\hat{H}^{(7)}\wedge\hat{C}^{(3)} + \text{ total derivative}\,. \tag{4.7}$$

After some algebra one obtains the following contribution to the four-dimensional action:

$$\mathcal{L}_{\mathrm{CS}} = -\tilde{H}^{(1)}C^{(3)} - \tilde{H}^{(2)\alpha}C^{(2)}{}_\alpha - \frac{1}{2!}\tilde{H}^{(3)\alpha_1\alpha_2}C^{(1)}{}_{\alpha_1\alpha_2} - \frac{1}{3!}\tilde{H}^{(4)\alpha_1\alpha_2\alpha_3}C^{(0)}{}_{\alpha_1\alpha_2\alpha_3}\,, \tag{4.8}$$

which after a partial integration gives

$$\begin{aligned}\mathcal{L}_{\mathrm{CS}} =& +\frac{1}{2}\tilde{B}^{(0)}G^IG^I - \tilde{B}^{(1)\alpha}G^I\bar{D}A^I{}_\alpha + \frac{1}{2}(G^I\mathcal{F}^I{}_{\alpha_1\alpha_2} - \bar{D}A^I{}_{\alpha_1}\bar{D}A^I{}_{\alpha_2})\tilde{B}^{(2)\alpha_1\alpha_2} - \\ &-\frac{1}{2}\tilde{B}^{(3)\alpha_1\alpha_2\alpha_3}\bar{D}A^I{}_{\alpha_1}\mathcal{F}^I{}_{\alpha_2\alpha_3} + \frac{1}{8}\tilde{B}^{(4)\alpha_1\ldots\alpha_4}\mathcal{F}^I{}_{\alpha_1\alpha_2}\mathcal{F}^I{}_{\alpha_3\alpha_4}\,.\end{aligned} \tag{4.9}$$

We now give the complete four-dimensional lagrangian with Yang-Mills fields, in the Einstein frame:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4\,, \tag{4.10a}$$

$$\begin{aligned}\mathcal{L}_1 =& \sqrt{-g}\bigg(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\mathrm{e}^{-\phi}\,F(V)_{\mu\nu}{}^\alpha G_{\alpha\beta}F(V)^{\mu\nu\,\beta} + \\ &+\frac{1}{4}D_\mu G_{\alpha\beta}D^\mu G^{\alpha\beta} - \mathrm{e}^\phi\,V_1(G)\bigg)\,,\end{aligned} \tag{4.10b}$$

$$\begin{aligned}\mathcal{L}_2 =& -\sqrt{-g}\bigg(\frac{1}{2}\mathrm{e}^{2\phi}\,\partial_\mu\tilde{B}^{(0)}\partial^\mu\tilde{B}^{(0)} + \frac{1}{4}\mathrm{e}^\phi\,\tilde{H}^{(2)}{}_{\mu\nu}{}^\alpha\tilde{H}^{(2)\mu\nu\,\beta}G_{\alpha\beta} + \\ &+\frac{1}{4!}\tilde{H}^{(3)}{}_{\mu\nu\lambda}{}^{\alpha_1\alpha_2}\tilde{H}^{(3)\mu\nu\lambda\beta_1\beta_2}G_{\alpha_1\beta_1}G_{\alpha_2\beta_2} + \\ &+\frac{1}{2\cdot 3!4!}\mathrm{e}^{-\phi}\,\tilde{H}^{(4)}{}_{\mu\nu\lambda\rho}{}^{\alpha_1\alpha_2\alpha_3}\tilde{H}^{(4)\mu\nu\lambda\rho\beta_1\beta_2\beta_3}G_{\alpha_1\beta_1}G_{\alpha_2\beta_2}G_{\alpha_3\beta_3}\bigg)\,,\end{aligned} \tag{4.10c}$$

$$\begin{aligned}\mathcal{L}_3 =& -\sqrt{-g}\,\mathrm{e}^{-\phi}\bigg(\frac{1}{4}G^I{}_{\mu\nu}G^{I\,\mu\nu} + \frac{1}{2}\mathrm{e}^\phi\,\bar{D}_\mu A^I{}_\alpha\bar{D}^\mu A^I{}_\beta G^{\alpha\beta} + \\ &+\frac{1}{4}\mathrm{e}^{2\phi}\,\mathcal{F}^I{}_{\alpha_1\alpha_2}\mathcal{F}^I{}_{\beta_1\beta_2}G^{\alpha_1\beta_1}G^{\alpha_2\beta_2}\bigg)\,,\end{aligned} \tag{4.10d}$$

$$\begin{aligned}\mathcal{L}_4 =& -\tilde{\epsilon}^{\mu\nu\lambda\rho}\bigg(\frac{1}{8}\tilde{B}^{(0)}G^I{}_{\mu\nu}G^I{}_{\lambda\rho} - \frac{1}{2}\tilde{B}^{(1)}{}_\mu{}^\alpha\,G^I{}_{\nu\lambda}\bar{D}_\rho A^I{}_\alpha + \\ &+\frac{1}{4}\tilde{B}^{(2)}{}_{\lambda\rho}{}^{\alpha\beta}\bigg(\frac{1}{2}G^I{}_{\mu\nu}\mathcal{F}^I{}_{\alpha\beta} - \bar{D}_\mu A^I{}_\alpha\bar{D}_\nu A^I{}_\beta\bigg) + \\ &-\frac{1}{2\cdot 3!}\tilde{B}^{(3)}{}_{\mu\nu\lambda}{}^{\alpha\beta\gamma}\bar{D}_\rho A^I{}_\alpha\mathcal{F}^I{}_{\beta\gamma} + \frac{1}{8\cdot 4!}\tilde{B}^{(4)}{}_{\mu\nu\lambda\rho}{}^{\alpha\beta\gamma\delta}\mathcal{F}^I{}_{\alpha\beta}\mathcal{F}^I{}_{\gamma\delta}\bigg)\,.\end{aligned} \tag{4.10e}$$

5. Scalar potential and extrema

In this section we will analyze the scalar potential. The potential consists of $V_1(G)$ (2.13) and terms involving the Yang-Mills scalars – the last term in (4.10d). In addition there are contributions from the vevs which can be introduced using $\tilde{H}^{(4)}$ – the last term in (4.10c), see sections 3.1 and 3.2. Note that in introducing these vevs we solve the equation of motion for certain components of $\tilde{B}^{(3)}$. These solutions should not be substituted back into the action, and therefore we will determine the potential from the equations of motion. To show that the same modified scalar potential appears in different equations of motion we work this out using an appropriate subset of fields from (4.10), namely the metric, the dilaton ϕ, the scalars $G_{\alpha\beta}$ and $\tilde{B}^{(3)}$. Later in this section we will investigate the complete equations of motion.

We therefore consider the following action (see also [20]):

$$\begin{aligned} S = \int \mathrm{d}^4x \sqrt{-g}\Big(& R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta} - \mathrm{e}^{\phi}\, V_1(G) - \\ & - \frac{1}{2\cdot 3!4!}\mathrm{e}^{-\phi}\, G_{\alpha_1\beta_1}G_{\alpha_2\beta_2}G_{\alpha_3\beta_3}\tilde{H}^{(4)}{}_{\mu_1\ldots\mu_4}{}^{\alpha_1\alpha_2\alpha_3}\,\tilde{H}^{(4)\beta_1\beta_2\beta_3\mu_1\ldots\mu_4}\Big)\,. \end{aligned}$$

The equations of motion read:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= -\frac{1}{4}\mathrm{tr}\,\partial_\mu G\partial_\nu G^{-1} + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left(\frac{1}{8}\mathrm{tr}\,\partial G\partial G^{-1} - \frac{1}{4}\partial\phi\partial\phi\right) + \\ &\quad + \frac{1}{2\cdot 3!3!}\mathrm{e}^{-\phi}\left((\tilde{H}^2)_{\mu\nu} - \frac{1}{8}g_{\mu\nu}\tilde{H}^2\right) - \frac{1}{2}g_{\mu\nu}\mathrm{e}^{\phi}\,V_1(G)\,, \quad (5.1a)\\ 0 &= \Box\phi - \mathrm{e}^{\phi}V_1(G) + \frac{1}{2\cdot 3!4!}\mathrm{e}^{-\phi}\tilde{H}^2\,, \quad (5.1b)\\ 0 &= \frac{1}{4}\big(G^{-1}(\Box G)G^{-1} - \Box G^{-1}\big)^{\alpha\beta} - \frac{1}{2\cdot 2!4!}\mathrm{e}^{-\phi}(\tilde{H}^2)^{\alpha\beta} - \\ &\quad - \mathrm{e}^{\phi}\frac{\partial}{\partial G_{\alpha\beta}}V_1(G)\,, \quad (5.1c)\\ 0 &= \nabla_{\mu_1}\big(\mathrm{e}^{-\phi}\,\tilde{H}^{(4)\mu_1\ldots\mu_4\,\alpha_1\alpha_2\alpha_3}G_{\alpha_1\beta_1}G_{\alpha_2\beta_2}G_{\alpha_3\beta_3}\big)\,. \quad (5.1d) \end{aligned}$$

We have used the abbreviations

$$\begin{aligned} \tilde{H}^2 &= G_{\alpha_1\beta_1}G_{\alpha_2\beta_2}G_{\alpha_3\beta_3}\tilde{H}^{(4)}{}_{\mu\nu\lambda\rho}{}^{\alpha_1\alpha_2\alpha_3}\,\tilde{H}^{(4)\mu\nu\lambda\rho\beta_1\beta_2\beta_3}\,,\\ (\tilde{H}^2)_{\mu\nu} &= G_{\alpha_1\beta_1}G_{\alpha_2\beta_2}G_{\alpha_3\beta_3}\tilde{H}^{(4)}{}_{\mu\sigma\lambda\rho}{}^{\alpha_1\alpha_2\alpha_3}\,\tilde{H}^{(4)}{}_{\nu}{}^{\sigma\lambda\rho\beta_1\beta_2\beta_3}\,, \quad (5.2)\\ (\tilde{H}^2)^{\alpha\beta} &= G_{\gamma_1\delta_1}G_{\gamma_2\delta_2}\tilde{H}^{(4)}{}_{\mu\nu\lambda\rho}{}^{\alpha\gamma_1\gamma_2}\,\tilde{H}^{(4)\mu\nu\lambda\rho\beta\delta_1\delta_2}\,. \end{aligned}$$

We now define $\mathcal{H}_{\alpha\beta\gamma}$ by

$$G_{\alpha_1\beta_1}G_{\alpha_2\beta_2}G_{\alpha_3\beta_3}\tilde{H}^{(4)}{}_{\mu_1\mu_2\mu_3\mu_4}{}^{\alpha_1\alpha_2\alpha_3} = \mathrm{e}^{\phi}\,\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\,\mathcal{H}_{\beta_1\beta_2\beta_3}\,. \quad (5.3)$$

The equation of motion (5.1d) implies that $\mathcal{H}_{\alpha\beta\gamma}$ is constant. The equations (5.1a)–(5.1c) can now be expressed in terms of

$$\mathcal{H}^2 \equiv \mathcal{H}_{\alpha\beta\gamma}\mathcal{H}_{\alpha'\beta'\gamma'}G^{\alpha\alpha'}G^{\beta\beta'}G^{\gamma\gamma'}\,. \quad (5.4)$$

The result is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{1}{4}\mathrm{tr}\,\partial_\mu G\partial_\nu G^{-1} + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left(\frac{1}{8}\mathrm{tr}\,\partial G\partial G^{-1} - \frac{1}{4}\partial\phi\partial\phi\right) -$$

$$-\frac{1}{4\cdot 3!}g_{\mu\nu}\,\mathrm{e}^{\phi}\,\mathcal{H}^2-\frac{1}{2}g_{\mu\nu}\,\mathrm{e}^{\phi}\,V_1(G)\,, \tag{5.5a}$$

$$0=\Box\phi-\mathrm{e}^{\phi}\,V_1(G)-\frac{1}{2\cdot 3!}\mathrm{e}^{\phi}\,\mathcal{H}^2\,, \tag{5.5b}$$

$$0=\frac{1}{4}\big(G^{-1}(\Box G)G^{-1}-\Box G^{-1}\big)^{\alpha\beta}-\mathrm{e}^{\phi}\frac{\partial}{\partial G_{\alpha\beta}}\bigg(\frac{1}{2\cdot 3!}\mathcal{H}^2+V_1(G)\bigg)\,. \tag{5.5c}$$

Here we use that

$$(\tilde{H})^{2\alpha\beta}=\frac{4!}{3}\,\mathrm{e}^{2\phi}\frac{\partial}{\partial G_{\alpha\beta}}\mathcal{H}^2\,. \tag{5.6}$$

Hence the potential has effectively changed to

$$V_{\text{eff}}(G)=\mathrm{e}^{\phi}\bigg(V_1(G)+\frac{1}{2\cdot 3!}\mathcal{H}^2\bigg)\,. \tag{5.7}$$

The complete scalar potential also includes the contributions from the Yang-Mills fields. But also the equations of motion of $\tilde{B}^{(3)}$ and $\tilde{B}^{(4)}$ will be modified by Yang-Mills contributions, and in case of $\tilde{B}^{(3)}$, the equation of motion contains additional terms due to the appearance of $\tilde{B}^{(3)}$ in $\tilde{H}^{(3)}$.

Let us continue the analysis for the specific case $\mathcal{G}=\mathrm{SU}(2)\times\mathrm{SU}(2)$. Then the $\tilde{B}^{(3)}$ contributions in $\tilde{H}^{(3)}$ lead to the 9 massive degrees of freedom given in table 2. These massive fields do not appear in the potential, so we do not consider them in this analysis. The complete equation of motion for $\tilde{B}^{(3)}$ reads:

$$\begin{aligned}&\frac{1}{12}\nabla_\rho\bigg(\mathrm{e}^{-\phi}\tilde{H}^{(4)\mu\nu\lambda\rho\,\beta\gamma\delta}G_{\alpha_1\beta}G_{\alpha_2\gamma}G_{\alpha_3\delta}\bigg)-\\ -\varepsilon^{\mu\nu\lambda\rho}\bar{D}_\rho A^I{}_{[\alpha_1}\mathcal{F}^I{}_{\alpha_2\alpha_3]}&+\tilde{H}^{(3)\mu\nu\lambda\,\beta\gamma}G_{\beta\delta}G_{\gamma[\alpha_1}f^\delta{}_{\alpha_2\alpha_3]}=0\,,\end{aligned} \tag{5.8}$$

while for $\tilde{B}^{(4)}$ we find

$$\frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\mathcal{F}^I{}_{[\alpha_1\alpha_2}\mathcal{F}^I{}_{\alpha_3\alpha_4]}+\tilde{H}^{(4)\mu\nu\lambda\rho}{}_{\beta[\alpha_1\alpha_2}f^\beta{}_{\alpha_3\alpha_4]}=0\,, \tag{5.9}$$

where the indices on $\tilde{H}^{(4)}$ have been lowered with $G_{\alpha\beta}$. Equation (5.9) can be rewritten with the use of (5.3):

$$\frac{1}{2}\mathcal{F}^I{}_{[\alpha_1\alpha_2}\mathcal{F}^I{}_{\alpha_3\alpha_4]}+\mathrm{e}^{\phi}\,\mathcal{H}_{\beta[\alpha_1\alpha_2}f^\beta{}_{\alpha_3\alpha_4]}=0\,. \tag{5.10}$$

Note that for $\mathcal{G}=\mathrm{SU}(2)\times\mathrm{SU}(2)$ (5.10) does not constrain $\mathcal{H}_{abc}$ or $\mathcal{H}_{\bar{a}\bar{b}\bar{c}}$. These are the fields for which we will turn on a nonzero vev, following section 3.2.

To proceed we make an additional simplification. We truncate the scalar sector to only the singlets of $\mathcal{G}$. That implies

$$G_{ab}=G_1\,\delta_{ab}\,,\qquad G_{\bar{a}\bar{b}}=G_2\,\delta_{\bar{a}\bar{b}}\,,\qquad G_{a\bar{a}}=0\,, \tag{5.11a}$$

$$A^I{}_\alpha=0\,. \tag{5.11b}$$

This truncation is well known in the analysis of scalar potentials: an extremum in the singlet sector is also an extremum of the potential without truncation [21]. The equations of motion (5.8) now simplifies, and becomes

$$\frac{1}{12}\nabla_\rho\big(\varepsilon^{\mu\nu\lambda\rho}\mathcal{H}_{\alpha_1\alpha_2\alpha_3}\big)+\tilde{H}^{(3)\mu\nu\lambda}{}_{\delta[\alpha_1}f^\delta{}_{\alpha_2\alpha_3]}=0\,. \tag{5.12}$$

For the choice $\alpha_1\alpha_2\alpha_3 = abc$ the second term in (5.12) vanishes and $\mathcal{H}_{abc}$, and similarly $\mathcal{H}_{\bar{a}\bar{b}\bar{c}}$, is constant. In this truncation we may therefore use the simplified analysis at the beginning of this section, and are again led to the potential (5.7).

In the analysis of the potential we write

$$\mathcal{H}_{abc} = \mathcal{H}_1\, \varepsilon_{abc}\,, \qquad \mathcal{H}_{\bar{a}\bar{b}\bar{c}} = \mathcal{H}_2\, \varepsilon_{\bar{a}\bar{b}\bar{c}}\,. \tag{5.13}$$

In terms of G_i and $\mathcal{H}_i$, $i = 1, 2$ we find

$$V_{\text{singlet}} = \mathrm{e}^{\phi}\bigg(-\frac{3}{2}\big[(G_1)^{-1} + (G_2)^{-1}\big] + \frac{1}{2}\big[(G_1)^{-3}(\mathcal{H}_1)^2 + (G_2)^{-3}(\mathcal{H}_2)^2\big]\bigg)\,. \tag{5.14}$$

The minimum value for fixed e^{ϕ} is at $G_1 = \mathcal{H}_1$, $G_2 = \mathcal{H}_2$ and reads

$$V_0 = -\mathrm{e}^{\phi}\left(\frac{1}{|\mathcal{H}_1|} + \frac{1}{|\mathcal{H}_2|}\right). \tag{5.15}$$

The potential in this extremum with respect to the G_i is a negative function of the dilaton.

6. Comparison with gauged $N = 4$ supergravity and conclusions

We performed the Scherk-Schwarz reduction of the 6-form version of $N = 1$, $d = 10$ supergravity. The result (4.10) contains a scalar potential that receives contributions of the form (5.7) from the vevs $\mathcal{H}_{\alpha\beta\gamma}$ of certain components of the 3-form fields $\tilde{B}^{(3)}$. In the reduction of the 2-form version the potential takes on a similar form [17]. There the $\mathcal{H}$ are constant fluxes of the 2-form along cycles of the internal space. We analyzed in particular the case $\mathcal{G} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and verified that, in the restriction to singlets of the gauge group, the potential (5.14) is identical to that of [17]. This potential has a minimum in the directions parametrized by G_1 and G_2. In contrast with the torus reduction — where the 6-form yields 15 massless scalars and 12 massless vectors (see appendix B.1) – the Scherk-Schwarz reduction over $\mathrm{SU}(2) \times \mathrm{SU}(2)$ yields only massive degrees of freedom: six 2-forms and nine 3-forms. This is also the case in the reduction of the ten-dimensional 2-form (see appendix B.3). The above is with the exception of the axion, which is massless in all these cases and does not appear in the scalar potential. The stabilization of the dilaton and the axion requires methods beyond the Scherk-Schwarz reduction used in this paper.

We now want to compare our results, in the absence of Yang-Mills fields, to those of gauged $N = 4$ supergravity in four dimensions. This comparison is facilitated by the analysis of [17], where the gauge groups associated with the Scherk-Schwarz and flux reductions were identified. In the absence of Yang-Mills fields there are 12 generators, satisfying the algebra $(i, j = 1, \dots, 6)$:

$$\begin{aligned} [T_i, T_j] &= f_{ij}{}^k T_k + \beta\, f_{ij}{}^k P_k\,, \\ [T_i, P_j] &= f_{ij}{}^k P_k\,, \\ [P_i, P_j] &= 0\,. \end{aligned} \tag{6.1}$$

For $\beta = 0$ this is the algebra associated with the Scherk-Schwarz reduction, β introduces the 3-form fluxes. For $i, j = 1, 2, 3$, and $f_{ij}{}^k = \varepsilon_{ijk}$ it corresponds to the group $\mathrm{CSO}(3, 0, 1)$ [22].

We start with the formulation discussed in [1, 2], where we limit ourselves to the case of gauged supergravity with six additional vector multiplets. This field content corresponds to that of the reduction discussed in section 2. The scalars in [1, 2] are the dilaton and axion in the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset, and scalars which parametrize an $\mathrm{SO}(6,6)/\mathrm{SO}(6) \times \mathrm{SO}(6)$ coset. These are expressed in terms of $Z_a{}^R$, $R = 1,\ldots,12$, $a = 1,\ldots,6$, with constraint $Z_a{}^R \eta_{RS} Z_b{}^S = -\delta_{ab}$. Here η_{RS} is the $\mathrm{SO}(6,6)$ invariant metric, chosen to be $\mathrm{diag}\,(-1,\ldots,-1,1,\ldots,1)$. The scalar potential reads

$$V_0 = \frac{1}{4}\mathrm{e}^{\phi}\Big(Z^{RU}Z^{SV}\big(\eta^{TW} + \frac{2}{3}Z^{TW}\big)\Big) f_{RST} f_{UVW}\,, \tag{6.2}$$

where $Z^{RS} = Z_a{}^R Z_a{}^S$ and $f_{RST} = \eta_{RU} f^U{}_{ST}$. Note that Z^{RS} is not an element of $\mathrm{SO}(6,6)$, but $W^{RS} = \eta^{RS} + 2Z^{RS}$ is.

To establish the precise correspondence between [17] and [1, 2], one has to identify the fields and structure constants. In [17] the $\mathrm{SO}(6,6)$ structure is described relative to the metric

$$L = \begin{pmatrix} 0 & \mathbb{1}_6 \\ \mathbb{1}_6 & 0 \end{pmatrix}. \tag{6.3}$$

The conversion to the metric η involves J:

$$J = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_6 & -\mathbb{1}_6 \\ -\mathbb{1}_6 & -\mathbb{1}_6 \end{pmatrix}, \qquad \eta = JLJ^{-1}\,. \tag{6.4}$$

Similarly, the scalar matrix M of [17] is related to $W = \eta + 2Z$ by $W = JMJ^{-1}$. The identification is (see e.g. [23])

$$Z_a{}^R = \frac{1}{2}\Big(E^{-1}(1+B) + E^T\ ,\ -E^{-1}(1-B) + E^T\Big)\,, \tag{6.5}$$

where E and B are 6×6 matrices. $G = EE^T$ are the scalars from the reduction of the metric and B the scalars from the 2-form. The structure constants in (6.1) should also be transformed by multiplication with J. For the gauge groups used in [17] one always (also when including the Yang-Mills contribution) has the property

$$\eta^{RU}\eta^{SV}\eta^{TW} f_{RST} f_{UVW} = 0\,. \tag{6.6}$$

Thus the results of [17] all fit into the formulation of [5].

We now choose the structure constants $f_{ij}{}^k$ in (6.1) to be those of $\mathrm{SU}(2)\times\mathrm{SU}(2)$. The potential (6.2), restricted to singlets ($B = 0$, G as in (5.11a)) then becomes

$$V_1 = \mathrm{e}^{\phi}(V_{11} + V_{22})\,, \tag{6.7}$$

where

$$V_{ii} = -\frac{1}{8}\big(3(G_i)^{-1} - (\beta_i)^2(G_i)^{-3}\big)\,. \tag{6.8}$$

The parameters β_i are the same as the flux parameter β in (6.1), we now have one of these for each factor of the gauge group. This is the same (up to a normalization) as the Scherk-Schwarz potential including the vevs $\mathcal{H}_i$, if we identify $\beta_i = \mathcal{H}_i$. We see therefore that this case can be reproduced directly by a gauging in four dimensions.

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However, these reductions do not give all gaugings in four dimensions. In the gauging of a direct product group $\mathcal{G}_{\text{gauge}}$ in $N=4$, $d=4$ supergravity, an $\mathrm{SU}(1,1)$ angular parameter α can be introduced for each factor of $\mathcal{G}_{\text{gauge}}$ [5]. These change in particular the dependence of the potential (6.2) on the dilaton and the axion. We considered this in detail for the gaugings of semisimple groups in [1, 2]. The Scherk-Schwarz reductions however yield non-semisimple gauge groups and it is interesting to see the effect of $\mathrm{SU}(1,1)$ angles on such gaugings. The reduction over $\mathrm{SU}(2)\times\mathrm{SU}(2)$ yields two copies of (6.1) with $f_{ij}{}^k=\varepsilon_{ijk}$, i.e. $\mathcal{G}_{\text{gauge}}=\mathrm{CSO}(3,0,1)\times\mathrm{CSO}(3,0,1)$. We can thus introduce two angles α_i and find the following potential (ℓ is the axion):

$$\begin{aligned}V_2 = \mathrm{e}^{\phi}\Big(&V_{11}(\ell\sin\alpha_1-\cos\alpha_1)^2+V_{22}(\ell\sin\alpha_2-\cos\alpha_2)^2+\\ &+\mathrm{e}^{-2\phi}(V_{11}\sin^2\alpha_1+V_{22}\sin^2\alpha_2)\Big)-\\ &-\frac{1}{4\sqrt{G_1G_2}}\sin(\alpha_1-\alpha_2)\left(3-\frac{\beta_1}{G_1}\right)\left(3-\frac{\beta_2}{G_2}\right).\end{aligned} \tag{6.9}$$

For $\alpha_1=\alpha_2=0$ this reduces to (6.7). For other values of the angles an extremum can be found for dilaton and axion if

$$\Delta = 4\sin^2(\alpha_1-\alpha_2)\,V_{11}V_{22}>0\,. \tag{6.10}$$

If this condition is satisfied the potential in the dilaton-axion extremum takes on the form

$$V_{2,\ell,\phi} = S\sqrt{\Delta}-\frac{1}{4\sqrt{G_1G_2}}\sin(\alpha_1-\alpha_2)\left(3-\frac{\beta_1}{G_1}\right)\left(3-\frac{\beta_2}{G_2}\right), \tag{6.11}$$

where S is the sign of $V_{11}\sin^2\alpha_1+V_{22}\sin^2\alpha_2$. This potential has an extremum at $G_1=\beta_1$, $G_2=\beta_2$, the same values as for (5.14). The value of the potential in this extremum is:

$$V_{\min} = \frac{1}{\sqrt{\beta_1\beta_2}}\left(\frac{1}{2}S\,|\sin(\alpha_1-\alpha_2)|-\sin(\alpha_1-\alpha_2)\right). \tag{6.12}$$

This can be positive or negative for appropriate choices of the angles α_1 and α_2. In the extremum the value of S is negative. The extremum is at a positive value of V for $\sin(\alpha_1-\alpha_2)<0$. In this case all second derivatives of the potential in the extremum vanish. Expanding around the extremum one finds that in the G_1 and G_2 directions the leading term is cubic function of G_i-2, while in mixed direction the potential is quartic and has a maximum. We emphasize the point that in contrast to (5.14) the potential (6.9) has an extremum in all variables.

In this paper we have restricted the analysis of the potentials to the singlet sector. This truncation has the property that extrema in these singlet variables are also extrema in the full theory. Nevertheless, the singlets do not determine the full structure, and it would be interesting to extend the analysis to all fields.

We have found that the results of the reduction of the $N=1$ $d=10$ theories in ten dimensions of [17] and of this paper are related to a gauged $N=4$ supergravity in four dimensions. In particular, the scalar potentials are the same. This does not necessarily

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imply that the four dimensional theories are equivalent. From the reduction we find massive two- and three-forms. Although the number of degrees of freedom is the same as in gauged $N = 4$ supergravity, the degrees of freedom are represented by different fields. It would be interesting to develop an $N = 4$ supergravity in terms of higher-form fields directly in four dimensions and to consider the possible gaugings.

The gauge group (6.1) is not semisimple, and therefore the analysis of [1, 2] does not apply. It will be interesting to systematically extend this analysis to more general groups. It remains an interesting problem to establish a relation between string theory and the $\mathrm{SU}(1,1)$ angles in $d = 4$.

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A. Notation and conventions

Our line element is $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ab}e^a \otimes e^b$ with $\eta_{ab} = \mathrm{diag}(-1, +1, +1, \ldots)$. The e^a are vielbein 1-forms, $e^a = e^a_\mu \mathrm{d}x^\mu$. The permutation symbol $\tilde{\varepsilon}$ is defined by

$$\tilde{\varepsilon}_{12\ldots D} = +1\,, \qquad \tilde{\varepsilon}^{12\ldots D} = -1\,. \tag{A.1}$$

The Levi-Cività tensor is defined by:

$$\varepsilon_{\mu_1\ldots\mu_D} = e\,\tilde{\varepsilon}_{\mu_1\ldots\mu_D}\,, \qquad \varepsilon^{\mu_1\ldots\mu_D} = e^{-1}\,\tilde{\varepsilon}^{\mu_1\ldots\mu_D}\,, \qquad \text{with } e^2 = -\det g\,. \tag{A.2}$$

p-forms $A^{(p)}$ are normalized as follows:

$$A^{(p)} = \frac{1}{p!}A_{\mu_1\ldots\mu_p}\,\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} = \frac{1}{p!}A_{a_1\ldots a_p}\,e^{a_1} \wedge \ldots \wedge e^{a_p}\,, \tag{A.3}$$

where

$$\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} = \mathrm{d}x^{\mu_1} \otimes \ldots \otimes \mathrm{d}x^{\mu_p} \pm \text{permutations}\,. \tag{A.4}$$

The exterior derivative d acts as:

$$(\mathrm{d}A)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]}\,. \tag{A.5}$$

The Hodge dual $\star A^{(p)}$ is defined by:

$$(\star A)_{\mu_1\ldots\mu_q} = \frac{1}{p!}\varepsilon_{\mu_1\ldots\mu_q}{}^{\nu_1\ldots\nu_p}A_{\nu_1\ldots\nu_p}\,, \tag{A.6}$$

where $q = D - p$. Thus $(\star)^2 = (-)^{pq-1}$ and

$$\star A^{(p)} \wedge B^{(p)} = \frac{1}{p!}A_{\mu_1\ldots\mu_p}B^{\mu_1\ldots\mu_p}\star\mathbb{1}\,, \tag{A.7}$$

where $\star\mathbb{1} = e\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^D$ is the volume form. We follow the conventions of [17] for the Riemann tensor.

B. Examples of the cohomology

B.1 6-Torus reduction of the 6-form

The 6-torus is a group manifold, $T^6 = \mathrm{U}(1)^6$, hence the analysis of section 3.1 can be applied. The cohomology of T^6 is generated by the left-invariant 1-forms $\mathrm{d}y^i \in \mathfrak{u}(1)^*$, $i = 1, \ldots, 6$. We have

$$\begin{aligned} b_p(T^6) &= \begin{pmatrix} 6 \\ p \end{pmatrix}, \\ \dim C^{(n)} &= 0\,, \qquad n = 0, \ldots, 6\,. \end{aligned} \tag{B.1}$$

There are no local shift symmetries, therefore only massless states will occur. There are 15 (scalar) states from $B^{(2)}$, 12 states from $B^{(1)}$ and one axion from $B^{(0)}$, as can be read off from table 1. The massless 2-forms can be dualized to scalars in four dimensions.

The number of possible vevs for $B^{(3)}$ is 20, which are precisely those in the span of $(\bigwedge^3 \mathrm{d}y^i)$.

B.2 $\mathcal{G} = \mathrm{SU}(2) \times \mathrm{U}(1)^3$ reduction of the 6-form

The Betti numbers of SU(2) and $\mathrm{U}(1)^3$ are given by

$$\big(b_0^{\mathrm{SU}(2)}, \ldots, b_3^{\mathrm{SU}(2)}\big) = (1,0,0,1)\,, \qquad \big(b_0^{\mathrm{U}(1)^3}, \ldots, b_3^{\mathrm{U}(1)^3}\big) = (1,3,3,1) \tag{B.2}$$

and we use Kunneth's formula to obtain the Betti numbers of $\mathcal{G} = \mathrm{SU}(2) \times \mathrm{U}(1)^3$:

$$b_r^{\mathcal{G}} = \sum_{n+m=r} b_m^{\mathrm{SU}(2)} b_n^{\mathrm{U}(1)^3}\,, \qquad (b_0^{\mathcal{G}}, \ldots, b_6^{\mathcal{G}}) = (1,3,3,2,3,3,1)\,. \tag{B.3}$$

The resulting degrees of freedom are given in table 3. Of course the total number of degrees of freedom, the sum of the numbers in the last column in table 3, is again 28.

There are 9 massive degrees of freedom in $B^{(3)}$. The two massless components in $B^{(3)}$ allow us to introduce vevs, they correspond to the 3-forms

$$\sigma^1\sigma^2\sigma^3\,, \qquad \text{and} \qquad \mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3\,, \tag{B.4}$$

where $\sigma^a \in \mathfrak{su}(2)^*$ and $\mathrm{d}y^i \in \mathfrak{u}(1)^*$. The massless components of $B^{(4)}$ do not appear in the action and do not give rise to vevs. There are 3 massive 2-forms, as well as 3 massless ones. The latter can be dualized to scalars and may contribute to the potential. There are three vectors that can be gauged away, corresponding to SU(2), and three massless physical vectors corresponding to $\mathrm{U}(1)^3$. The single scalar $B^{(0)}$ is the axion.

B.3 $\mathcal{G} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ reduction of the 2-form

Another interesting example is the reduction of a 2-form field $\hat{B}^{(2)}$ from 10 to 4 dimensions [17, 23]. In this case we obtain 2-, 1-, and 0-forms: $B^{(2)}$, ${B^{(1)}}_\alpha$ and ${B^{(0)}}_{\alpha\beta}$ respectively. We consider $\mathcal{G} = \mathrm{SU}(2) \times \mathrm{SU}(2)$, of which the relevant Betti numbers are

$$(b_0, \ldots, b_3) = (1,0,0,2)\,. \tag{B.5}$$

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field	features	dimension	# DOF	total DOF
$B^{(4)}$	gauge	3	0	0
	massive	9	0	0
	massless	3	0	0
$B^{(3)}$	gauge	9	0	0
	massive	9	1	9
	massless	2	0	0
$B^{(2)}$	gauge	9	0	0
	massive	3	3	9
	massless	3	1	3
$B^{(1)}$	gauge	3	0	0
	massive	0	3	0
	massless	3	2	6
$B^{(0)}$	gauge	0	0	0
	massive	0	0	0
	massless	1	1	1

Table 3: Result of the analysis of the Scherk-Schwarz gauge transformations of the Kalb-Ramond fields $B^{(n)}$ for the group $SU(2) \times U(1)^3$, for the reduction of $\hat{B}^{(6)}$ from 10 to 4 dimensions. See the caption of table 1 and 2 for more details.

field	features	dimension	# DOF	total DOF
$B^{(2)}$	gauge	0	0	0
	massive	0	0	0
	massless	1	1	1
$B^{(1)}$	gauge	0	0	0
	massive	6	3	18
	massless	0	2	0
$B^{(0)}$	gauge	6	0	0
	massive	9	1	9
	massless	0	1	0

Table 4: Result of the analysis of the Scherk-Schwarz gauge transformations of the Kalb-Ramond fields $B^{(n)}$ for the group $SU(2) \times SU(2)$, for the reduction of $\hat{B}^{(2)}$ from 10 to 4 dimensions. See the caption of table 1 and 2 for more details.

The dimensions of the spaces $C^{(p)}$ are $\dim C^{(0)} = \dim C^{(1)} = 0$, $\dim C^{(2)} = 6$, $\dim C^{(3)} = 9$. The resulting degrees of freedom are given in table 4. There is a massless 2-form (which can be dualized to an axionic scalar), there are six massive vectors, and nine massive scalars, giving again a total of 28 degrees of freedom.

These are the same number of degrees of freedom as for the 6-form case (table 2): the massless now appears as a two-form, the 9 degrees of freedom represented by 3-forms in table 2 now appear as scalars, the massive 2-forms from table 2 now appear as vectors.

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For the T^6 reduction of the 2-form version we have $(b_0, b_1, b_2) = (1, 6, 15)$, giving massless degrees of freedom only: 1 for $B^{(2)}$ (the axion), 6 massless vectors for $B^{(1)}$, and 15 massless scalars for $B^{(0)}$. Again these are the same degrees of freedom as obtained from a reduction of the 6-form (see appendix B.1).

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